What is a straight line? Does this matter in mechanical analysis?

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Abstract: It is an assumption that engineers make every day: we draw a straight line to define our product geometry, and then we analyze the product performance – but do we ever challenge the reality of the geometry and how that would influence the performance? Small and smooth variations to a product geometry are generally anticipated in the design drawing tolerance definitions, and for a simple product or component these can be taken into account in analysis by assuming the maximum or minimum dimensions. For a more complex component, containing multiple load paths, it is not necessarily the case that choosing to analyze the maximum or minimum dimension cases would identify the strongest or weakest component within the tolerance range. This becomes further complicated when considering non-smooth geometry. How is lack of smoothness defined and recognized, and what kinds of influence does it have on the performance of the component? For example, component inspection might identify a small geometrical defect triggering a concession analysis of the stress raiser feature. On a more systematic level, should we not have a more systematic approach? Non-smooth geometry can be represented in mathematics by fractals and wavelets. In this paper I will explore non-smooth surface effects on the surface of an axi-symmetric uni-axial test specimen, considering how the size of the non-smooth surface features disturb the nominal stress state, and the development of localized plasticity. The analysis will also include fatigue cycling.

Keywords: Elasticity, Fatigue, Fractals, Geometry, Plasticity, Residual Stress.

1. Introduction

In the stress analysis of an engineering component, the stress engineer considers the geometry of that component. The design drawings would probably indicate a nominal geometry, and tolerance, and possibly also provide some information about the manufacturing process. Modern design drawings would almost certainly be available in electronic format, which the stress engineer could import directly as the basis for the stress analysis model. This instantly raises the question: should the analysis be based on the nominal geometry or the worst tolerance, and what difference would that make? The macroscopic form of the surface could vary in a smooth way between values determined by the minimum and maximum tolerance: what difference does that imply to the stress analysis result? Another consideration for the stress engineer is the extent to which component features such as fillet radii, bolt holes, and so on, need to be captured explicitly within the geometric representation of the stress analysis model.
A similar consideration is that of the exact location and representation of the load (Saint-Venant, 1885), explained by Love as “…the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load” (Love, 1944). It is obvious that small changes in geometry will have an impact on the load path and load distribution, so that the Saint-Venant principle is also applied to geometry. Typically, where the Saint-Venant principle is invoked, the geometry is continued about three times the distance from the source of variation as is the geometrical size of the variation. In more recent times, this type of question has been discussed in detail by a number of authors (Ainsworth, 2000; Armstrong, 2000). This is also the basis for engineering detail design, whereby localized regions of a component are modified to improve performance (Robinson, 2012).

The above works assume a smooth, albeit imperfect, surface boundary, and the Finite Element Analysis (FEA) stress analysis technique. Boundary Element Method (BEM), although currently not commonly used by stress engineers, remains a viable option for many types of analysis. Interestingly, unlike FEA which requires computation on the network of elements and nodes throughout the entire material domain of the model, the BEM linear elastic solution is predicated on a representation of the boundary only. Stress, deflection and other parameters within the domain can be computed after the solution, so only those regions of particular interest need to be computed. The point of interest to the current discussion is that although BEM is capable of producing very accurate results on the nominal material boundary, and well within the material domain, the results it provides for regions close to the boundary are not reliable (McMillan, 1988).

Now let us consider the process of manufacture of engineering components and the implications that has for real surface geometry. Machining techniques (turning, drilling, milling, etc.) leave a surface profile with fairly regular special frequency features on the length scale of fractions to tens of micrometers (Bowden, 1964). Polishing techniques can reduce the length scale of such features to the order of tens of nanometers (Yu, 2011). For length scales smaller than this, classical solid mechanics must give way to computation on the molecular level (Clark, 2005).

The example presented in this paper is concerned with the axi-symmetric Finite Element Analysis of a uni-axial test specimen. While it might be fair to invoke the Saint-Venant principle in respect of smoothly varying surface geometry, it is clear that for realistic fine detail surface roughness there will be significant stress-raising features, leading to localized plasticity (Hill, 1998). It is the presence of the plasticity that makes the problem interesting, and applying a sequence of tensile and compressive loads illustrates the fatigue process.

Other mechanics problems where including surface roughness in the analysis model could lead to a significantly different analysis prediction include: contact mechanics, plate vibration, stress wave propagation close to a rough surface and laminar flow.

The non-smooth contact mechanics problem is a non-trivial problem: the analytical solution to contact problems was found by Hertz (Hertz, 1881; Johnson, 1989), but it makes particular assumptions about surface curvatures and depth of material. In elastic contact theory, a given load leads to the formation of a contact area. Small surface flaws might lie within or near to the edge of the contact area, so under varying contact load, the stress state around the surface flaw changes significantly. Understanding this better would lead to better predictions of edge of bedding fatigue failure in bearings and other contacting component systems.

In the case of plate vibration frequencies and mode shapes, the question of frequency prediction is an interesting one. There has been a long standing open mathematical question, “Can one hear the
shape of a drum?” (Kac, 1966). This question has been answered fairly recently in the negative, with a number of counter examples, but Kac’s analysis shows that both area and perimeter length can (in theory) be deduced from the eigen-spectrum of a vibrating system, and the construction of geometry given a part of the eigen-spectrum has been considered (Colin de Verdiere, 1987). The use of optimization methods to influence spectral densities in different parts of the spectrum range has been shown, albeit for point mass addition rather than perimeter modification (McMillan, 1996, 1997). More recently, there has been an investigation on vibration frequencies and mode shapes of plates with fractal boundaries (Tee, 2015), and of plates where the perimeter is perforated (Chechkin, 2011).

In classical wave mechanics (Kolsky, 1963; Achenbach, 1990), stress waves have been shown to have emergent properties when they reach and interfere at a boundary, or within a bounded medium, to form Rayleigh, Stoneley and Lamb waves (Rayleigh, 1887; Lamb, 1904). With the computational power now available, it is feasible to consider modelling wave propagation in modest sized domains using Explicit Finite Element (McMillan, 2013; McMillan 2015), and this opens up the possibility to consider the effect of surface roughness on the formation and morphology of such surface and wave-guided wave forms and frequencies.

A final class of analysis problems for which surface roughness could be expected to play a significant part is that of nominally laminar fluid flow over a rough surface. The field of classical fluid dynamics (Batchelor, 1983) shows simple analytical solutions for Poiseuille flow through a pipe and shear driven flow between two parallel planes, known as Couette flow. On-set of turbulence is determined by Reynold’s number, which is proportional to the product of flow velocity and typical length scales of features in the flow domain, and inversely proportional to viscosity. It is known that a disturbance to the flow, caused by a localized geometry deviation, will die away downstream, so long as flow velocity is sufficiently low or viscosity is sufficiently high: this is what is meant by “laminar flow”. For the case of flow in a rough pipe or over a rough surface, the size of the roughness features should now be considered to be the “typical length scale”, meaning that the effective Reynold’s number in the vicinity of the rough surface is significantly lower than that in the mid-stream. Since all real surfaces have some level of roughness, this means that flow sufficiently near a wall will always be turbulent. With a sufficiently fine CFD grid, this turbulence should be possible to model and illustrate using a laminar flow formulation.

2. The problem of representing surface roughness

While the concept of “roughness” is understood by even the youngest child, it is not a simple matter to describe and characterize it in such a way that geometry can be produced that is both representative of real geometry and suitable for Finite Element analysis. The representation of 3D geometry presents many complications (de Berg, 2008): in this present uni-axial case study, the geometric modeling requirement is for the 2D form.

With regard to Finite Element representation of the material domain, the underlying question is whether to attempt to capture the geometric features to the highest resolution possible, or to define a limit size and model the domain with equal sized “pixel” (2D) or “voxel” (3D) elements. Since the minimum size of a surface roughness has no sensible lower size bound, there is no sensible limit to the resolution requirements for a representative mesh – there will always be a length scale
at which the surface form implied by the element node distribution is jagged enough to attract high local stress concentrations. The approach adopted here is to accept the limitation of a minimum element size, and ensure that the effect of this is uniform by using a regular “pixel” mesh within the region of interest within the domain. To reduce the local stress concentration effects quadratic rather than linear element formulations were employed: note that the region of interest lies not at the surface, but within the domain close to the surface. Unrealistic local stresses can be ignored, and the arising local elastoplastic strains rebalance the load distributions into the domain. Of course, the argument is recursive – modelling at higher resolution is simply a matter of changing a length scale factor: whether this is satisfactory or useful is a matter for the reader to judge.

2.1 Wavelets and Fractals

Wavelets (Haar, 1910; Goswami, 1999; Press, 2007) and fractals (Mandelbrot, 1982) are two inter-related mathematical topics, both of which define multiply discontinuous functions, and which could be used for the basis of a systematic mathematical model of material surface roughness. Another mathematical topic of interest is percolation theory (Sahimi, 2009), which has been applied to the modelling of the process of corrosion. An important property of wavelets is their orthogonality, which is a fundamental assumption necessary for computational algorithms in classical mechanics (Young, 1990).

It is the nature of fractals to be self-similar at different length-scales, and a class of fractals can be constructed by taking a shape at one length-scale and replicating and re-scaling that shape as a modification detail onto the original shape. Such a constructed fractal is very similar to a wavelet.

![Figure 1. Three representations of surface roughness.](image)

2.2 Fractal shapes applied to finite element geometry

In this study, because a “pixel” mesh grid is required, a fractal family that can be meshed by a square grid is desirable, and such an example is the Koch curve (type 2), also known as the...
Minkowski sausage. The construction and grid mesh of this family is shown in Figure 1. The basic form is shown in Figure 1(a), replicated and scaled in Figure 1(b), and then that form is reapplied to the original, shown in Figure 1(c).

An important measure in fractals is the Hausdorff dimension, which provides a measure of surface length to enclosed area. For the Minkowski sausage the Hausdorff dimension is 1.5.

3. Computational example: axisymmetric test specimen

The case considered here is that of a simple axisymmetric axial test specimen. Typically, for such a test specimen, there is a narrow gauge section, where, under tensile or compressive loading, the material should be placed under a high uni-axial state of stress. Close attention is given to the surface finish of the test specimen in this gauge section, as surface roughness is known to have a significant influence on the test result. The size and type of influence of surface roughness is also explored.

3.1 Configuration, material properties and load case

For the purpose of this exercise, the gauge section is taken to be of diameter 4 mm, and invoking the Saint-Venant principle (Love, 1944), it is deemed necessary only to model a section of just 1 mm thickness. In order to avoid geometry difficulties at the top and bottom of the model, the development of surface roughness is restricted to a central band of thickness 80 μm. Because of the length scales of interest, it is difficult to show graphically as an abaqus model, and instead a schematic is shown in Figure 2.

![Figure 2. Axisymmetric test-piece geometry.](image)

The material properties taken are those for steel, with data taken from a simple training example given in the Abaqus manuals, Table 1. The plasticity definition is illustrated in Figure 3.
Table 1. Material data.

<table>
<thead>
<tr>
<th>Elastic properties</th>
<th>Plastic properties</th>
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</thead>
<tbody>
<tr>
<td>Young’s Modulus (GPa)</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>210</td>
<td>0.3</td>
</tr>
<tr>
<td>350</td>
<td>0.025</td>
</tr>
<tr>
<td>375</td>
<td>0.1</td>
</tr>
<tr>
<td>394</td>
<td>0.2</td>
</tr>
<tr>
<td>400</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Figure 3. Plasticity definition.

The minimal boundary conditions required rigid body motion control are applied. The load is applied as a pressure load, which, for a tensile test, is $-270$ MPa. Note that this is 90% of the stress required for the onset of plasticity, and would normally be considered to be reasonably well within the elastic regime.

### 3.2 Nominal geometry

For nominal geometry case, there being no surface roughness, the state of stress in the gauge section is uniformly and identically that of the applied pressure load. This was checked in Abaqus, using meshing schemes similar to the complex meshing requirements for the rough surface models.
3.3 Three styles of surface roughness

In the initial investigation, three styles of surface roughness were investigated, corresponding to the shapes shown in Figure 1. Comparison of styles (a) and (b) explores the length-scale and mesh-scale effects, while comparison of these with style (c) explores the notion of roughness complexity. In all cases the array of roughness for each test was uniform: in cases (a) and (c) the repeating shape is of length $4 \mu m$, so that 20 repeating shapes filled the space of $80 \mu m$, and for case (b) the length was $1 \mu m$, requiring 80 repeating shapes. Figure 4 illustrates the sketch construction method employed to create these shapes.

![Construction of the fractal edge.](image)

**Figure 4.** Construction of the fractal edge.

![Meshing strategy.](image)

**Figure 5.** Meshing strategy.
Figure 6. Three styles of surface roughness – von-Mises: upper row – middle of rough section; lower row – lower end of rough section.
Figure 7. Three styles of surface roughness – PEEQ: upper row – middle of rough section; lower row – lower end of rough section.
Meshing was somewhat complicated, since very fine elements were required to capture the roughness geometry detail, but at a distance from the roughness region, the element size was not important. The basic meshing strategy adopted is illustrated in Figure 5. In detail, the global seed size was set to 20 µm, and edge seed sizes in the regions around the roughness region were set to 0.25 µm. A certain amount of effort in partitioning, and the use of Medial axis meshing was necessary to ensure the regular grid of elements at the roughness region. Mesh type was CAX8R / CAX6 (Modified formulation unticked).

For each surface roughness style two results are presented. The upper figures show the results near the center of the roughness region. The lower figures give an impression of the edge effect, moving from a perfectly smooth (and therefore unrealistic) surface to a rough one.

The stress analysis results for these three cases are presented in Figure 6 (von-Mises stress) and Figure 7 (PEEQ). Note that in the presentation of the von-Mises stress, a “User defined” contour interval has been employed, which corresponds to: zero stress; the nominal stress, 270 MPa; onset of plasticity at 300 Pa; and the plasticity definition intervals as defined in Table 1. The PEEQ results are displayed using a Log scale.

Figure 8. Large defect von Mises stress on five load reversals.
4. **Axisymmetric specimen under repeated loading**

Following on from the results above, it is clear that a sizable plastic zone is created. This plastic zone extends well beyond the surface elements, and this provides a counter argument regarding how representative such a mesh might be for this particular type of geometry with so many internal corners. Once the plastic zone exists, the question arises as to how it would be developed on subsequent loadings. This was examined by applying further load steps; 1\(^{st}\) tension: to apply a nominal tensile stress of −270 MPa, 1\(^{st}\) compression: a load reversal to apply a nominal compressive stress of 270 Pa, 2\(^{nd}\) tension: another reversal to give a nominal tensile stress at −270 MPa, 2\(^{nd}\) compression: a nominal compressive stress of 270 MPa, and finally 3\(^{rd}\) tension: a nominal tensile stress of −270 MPa.

![Figure 9. Large defect PEEQ on five load reversals.](image)

4.1 **Specimen with regular array of five simple defects**

In this case the roughness feature was scaled up to a size of 16 \(\mu\)m, to give a repeating set of five within the 80 \(\mu\)m roughness region. A similar meshing approach was applied, but in this case great care was taken to ensure that for the whole area of interest (regions where von-Mises and 2015 SIMULIA UK Regional User Meeting 11
PEEQ are displaying significant values) the mesh is a uniform regular grid of 0.25 μm square elements.

As the mesh is fine, and regular, it is not shown in Figures 8 and 9. The results for von-Mises stress (Figure 8) show very little obvious change with increasing number of load reversals, but a careful examination of the green ligatures between internal corners reveals a tendency for these to join up and thicken. The results for PEEQ are more revealing (Figure 9). The outer boundary at 1×10⁻⁷ strain remains fairly constant. Plotting at smaller, non-zero values for PEEQ did not discernibly extend this boundary, but this is a matter for more careful research in future. Within the outer boundary, the amount of plastic strain increases significantly with load reversals. Notice that the plastic strain region extends into the material to a fairly fixed depth of around the size of the roughness feature.

4.2 Specimen with complex roughness at a large irregular defect

The next case to examine is that of a uniform but more complex surface roughness, based on the feature style described in Section 3 as style (c). Added to that definition, is a larger defect (feature length 16 μm), which is modified to resemble style (c) – a third order Minkowski sausage geometry. The same 0.25 μm regular grid meshing strategy was adhered to as described above.

Figure 10. Irregular large defect: von-Mises stress at first and fifth load reversals.
The same five load reversals were applied, but here the results are shown only for the first and last. Figure 10 shows the von-Mises stress in the region of the larger central defect. Notice that the pale blue color now represents stresses exceeding 300 MPa, and that this area grows significantly from the first applied load, to the fifth load reversal. Also note that the stress levels in the vicinity of the deepest part of the notch are showing heightened levels of stress, even away from the elements at the corner. Figure 11 shows the PEEK strain, which displays a similar constant boundary as seen in Figure 9, and levels of plasticity growing within this boundary.

5. Discussion and conclusions

This very simple case demonstrates some interesting issues surrounding surface roughness effects in tensile test specimens.

For a “regularly” rough surface, the height of the major undulation provides a measure as to the depth of the effect. Plasticity will be expected within that depth.
For an “irregularly” rough surface, the height of the worst undulation provides a measure of its effect on the overall performance, but this height would typically include a peak (outward of the nominal surface) and a trough (inward of the nominal surface). In real life, the outward part might be smoothed off, but it should still be counted as part of the roughness feature dimension.

The analysis presented here is for geometry in the form of a Minkowski sausage, which has Hausdorff dimension 1.5. The Hausdorff dimension is a significant measure of the fractal length dimension, and it would be important to assess real surfaces to see whether their Hausdorff dimension is similar, and whether a more realistic fractal model is required for future computational simulations of roughness.

There remain many unanswered questions about meshing and mesh quality, and concerning results at high numbers of load reversals.

Further investigation employing fracture mechanics using XFEM would be of interest.

6. References


7. Acknowledgements

I would like to acknowledge the continued support of colleagues and students at Glyndwr University.

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