A Second-Invariant Extension of the Marlow Model: Representing Tension and Compression Data Exactly

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Abstract: *The deviatoric response of the Marlow model in Abaqus is defined by test data from a single mode of deformation. For example, the model can be defined with uniaxial tension data or uniaxial compression data but not with both (except in very limited cases involving 1-D elements). The fact that the Marlow model depends only on the first strain invariant causes the limitation. This paper shows that the Marlow model in Abaqus is a special case of a hyperelastic potential function which involves the first and second strain invariants. The extended Marlow model is defined by uniaxial tension and equibiaxial tension data (or, equivalently, uniaxial compression data). Furthermore, the extended model will exactly reproduce the entire set of test data used to define it, just as the first-invariant model does. An implementation of the model in a UHYPER subroutine is discussed and results from an analysis of a component are presented.*

Keywords: *Constitutive Model, Elastomer, Hyperelasticity, Polymer, Rubber*

1. Introduction

The constitutive behavior of elastomers and some polymers is often modeled using hyperelasticity. The behavior of an isotropic hyperelastic material is defined by a strain energy density function which relates strain invariants to the amount of energy in the material. Many forms of the strain energy function have been programmed into the Abaqus code. All of them except the Marlow model have a parameterized functional form. In applications, the parameters are determined by curve-fitting data from tests in fundamental modes of deformation.

The Marlow model for hyperelasticity in Abaqus is completely defined by stress-strain data from a test in a single mode of deformation such as uniaxial tension. In simulations, the model will faithfully reproduce the test data used to define it. However, test data from another mode of deformation, for example uniaxial compression, cannot be used to alter the definition. This limitation arises from the fact that the Marlow model depends only on the first strain invariant (Marlow, 2003). Except in limited cases involving 1-D elements, test data above and beyond that used to define the model is useful only for comparison to the simulated mechanical responses.

The first-invariant Marlow model in Abaqus is a special case of a more general relation involving the second strain invariant. The more general relation is capable of representing the mechanical response of the material to additional deformation modes. It can, for example, reproduce the mechanical response to uniaxial tension *and* uniaxial compression. The more general relation, which involves the second strain invariant as well as the first, has been known to the author since the time when the first-invariant Marlow model was introduced into Abaqus. The motivation of this paper is the desire to publicize the existence of a two-invariant Marlow model that can capture the mechanical response of a material in multiple, fundamental deformation modes.

This paper describes the basic mathematical form of the two-invariant Marlow model and its implementation in a UHYPER user subroutine. The UHYPER subroutine is outlined and some results are presented for a comparison of the two-invariant Marlow model with the Ogden model.

2. Mathematical Form of the Two-Invariant Marlow Model

A strain-energy density function for an incompressible, two-invariant Marlow model is shown in Equation 1. The first invariant, I_1, is the trace of the left Cauchy-Green deformation tensor B. The second invariant in the equation, I_2, is not the traditional second invariant. Rather, it is the trace of B^2. The other terms in Equation 1 are described in the following paragraphs. Equation 1 is written in terms of data from a uniaxial tension test and an equibiaxial tension test. Similar relations can be developed which include test data in other deformation modes. The number of independent tests is not limited to 2 but the complexity of the relation increases substantially as more deformation modes are included.

$$U(I_1, I_2) = C_u(I_1, I_2)\int_1^{f(I_1)} T_u(\lambda)d\lambda + C_e(I_1, I_2)\int_1^{h(I_1)} 2T_e(\lambda)d\lambda$$

Equation 1. A Two-Invariant Marlow Model

The function f returns the uniaxial stretch at which the first invariant has a given value I_1. The uniaxial response function T_u represents the uniaxial tension response of the material derived from test data. That is, $T_u(\lambda)$ is the nominal uniaxial stress at the uniaxial stretch λ. The first integral is the energy caused by a uniaxial deformation to a stretch of $f(I_1)$.

The function h returns the equibiaxial stretch at which the first invariant has a given value I_1. The equibiaxial response function T_e represents the equibiaxial tension response of the material derived from test data. That is, $T_e(\lambda)$ is the nominal equibiaxial stress at the equibiaxial stretch λ. The second integral is the energy caused by an equibiaxial deformation to a stretch of $h(I_1)$.

The partitioning functions C_u and C_e are the key factors which allow the two-invariant model to represent precisely more than one deformation mode. The partitioning functions sum exactly to 1 for all allowable deformations. The first function, C_u, is 0 for a tensile equibiaxial deformation while the second function, C_e, is 0 for a tensile uniaxial deformation. The first-invariant Marlow model can be recovered by discarding the second term in Equation 1 and setting C_u to 1. Natural, closed-form representations for the partitioning functions and for the functions f and h exist. These are used in a UHYPER subroutine outlined in the following section but are not defined here.

3. Outline of UHYPER Subroutine for the Two-Invariant Model

The UHYPER subroutine (Dassault Systèmes, 2007) allows a user to define a hyperelastic material in terms of derivatives of the strain density. Equation 1 has been implemented in a UHYPER subroutine for a particular set of material test data.

The implementation of Equation 1 is relatively straight-forward. First, Equation 1 is differentiated as required by the UHYPER subroutine. The differentiation must account for the nonstandard definition of the second strain invariant. The integrands are assumed to be “nice” functions when computing the derivatives. Next, cubic-spline representations of T_u and T_e are defined using a particular set of test data and are programmed into subroutines which are called by UHYPER when necessary. Finally, the UHYPER subroutine is programmed to compute the derivatives of the strain-energy density function. Compressibility can be introduced by defining derivatives with respect to the volume ratio. Compressibility will cause a slight deviation of the simulated responses from the test data. The deviation will not be a problem in applications involving nearly incompressible materials.

The simulated uniaxial response for a 1-element model is compared with the test data in Figure 1. The simulated response passes through the test data just as the first-invariant Marlow model would. Results from a third-order Ogden model are shown for comparison. The Ogden model is defined by a curve-fit to a complete set of test data (uniaxial tension, equibiaxial tension, planar tension).

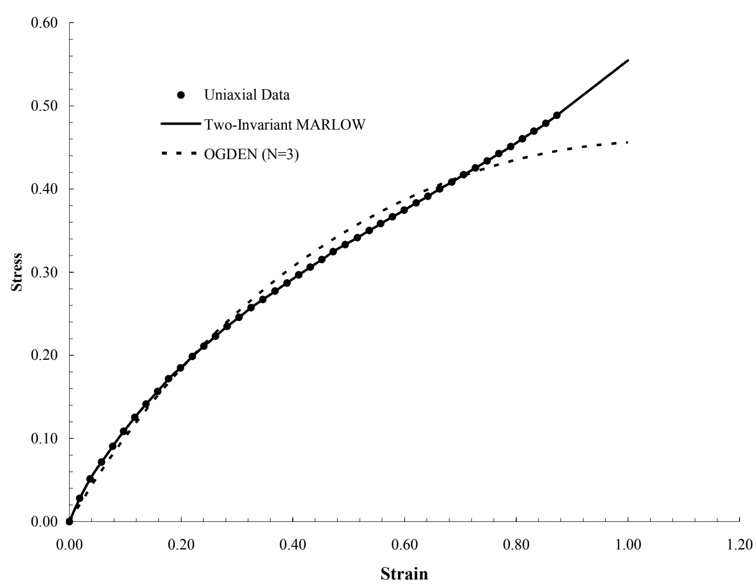

Figure 1. Simulated Uniaxial Response

Figure 1 shows a simulated response for the two-invariant Marlow model beyond the end of the test data. An extrapolated response is created in a natural way by defining an extrapolation of the uniaxial response function.

Figure 2 shows a comparison of the simulated equibiaxial response to the test data. The simulated response passes through the test data just as is it should. Again, an extrapolated response can be included in a natural way.

The two-invariant Marlow model in Equation 1 is completely defined by the uniaxial and equibiaxial tension data. Test data for planar tension does exist for the material represented by the UHYPER subroutine. The simulated planar tension response is compared to the test data in Figure 3. We should not expect the simulated response to pass precisely through this data because it was not included in the definition of the UHYPER subroutine. However, in this case, the simulated response is very close to the test data.

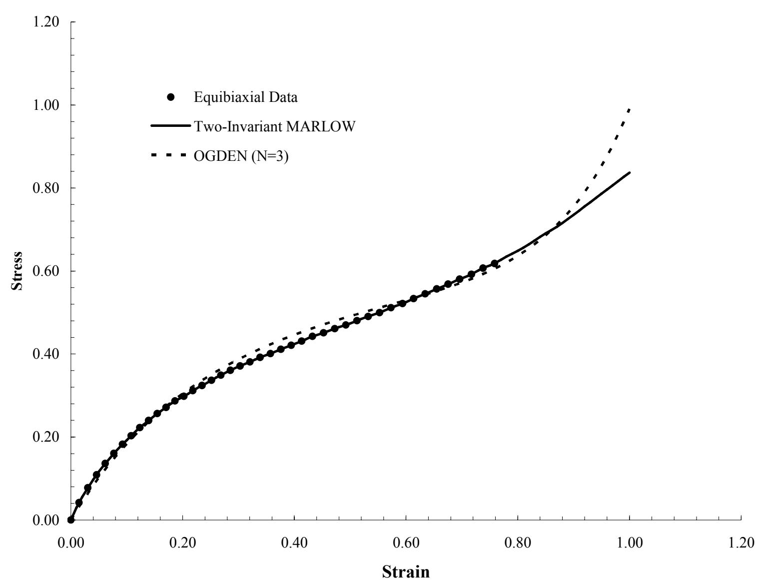

Figure 2. Simulated Equibiaxial Response

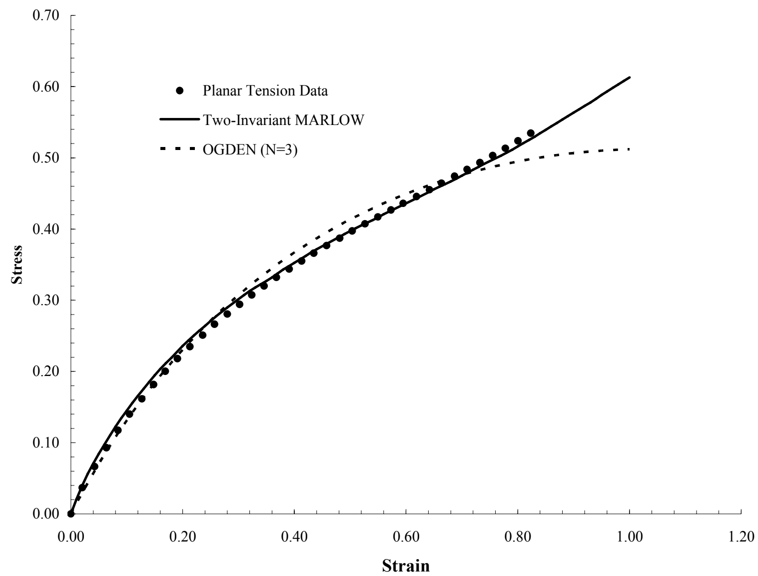

Figure 3. Simulated Planar Tension Response

4. A Results Comparison

The UHYPER subroutine described above has been used in models that are much more complicated than the simple 1-element simulations used for the simple deformation mode analyses. For example, the UHYPER subroutine has been used in the shifter boot model shown in Figure 4. The shaft is rotated and the boot deforms as shown.

Based on Figures 1 through 3, we expect results generated by the model with the UHYPER subroutine to be very similar to those generated using the third-order Ogden model. This is indeed the case. Figure 5 shows the maximum principal logarithmic strain in the boot at the end of the calculations. The strains, as well as the deformed shapes, are very similar. The reaction moments enforcing the rotations differ by less than 3%. The calculations were done in Abaqus 6.7-4.

The UHYPER subroutine is not as computationally efficient as the Ogden model. This is attributed to the more complicated functional form of the two-invariant Marlow model and the fact that is uses cubic splines. Still, the performance of the UHYPER subroutine is not unacceptable. A performance comparison is shown in Table 1. Note that run times are normalized by the run time for the Ogden model.

The performance of the UHYPER subroutine for other applications has been found to be similar. In some cases, the two-invariant Marlow model allows a solution to continue past a point where it failed with an Ogden model because the Ogden model became unstable.

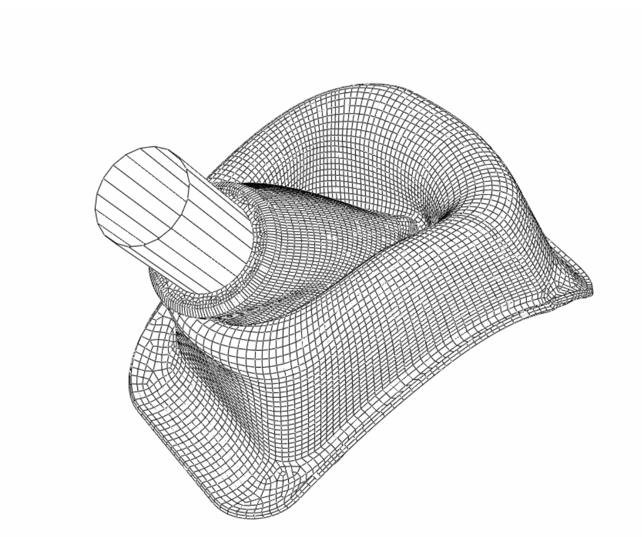

Figure 4. Deformed Shifter Boot Finite-Element Model

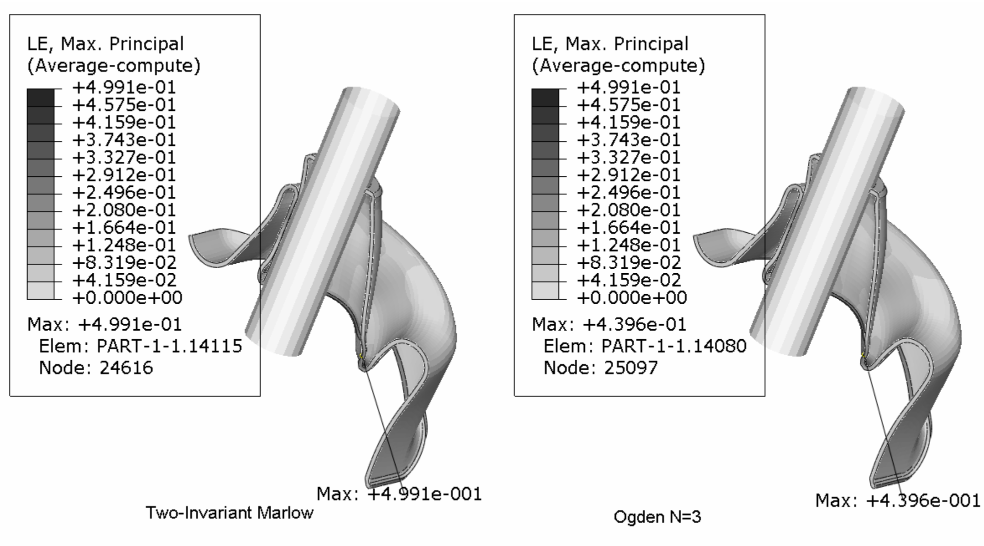

Figure 5. Maximum Principal Log Strain

Model	Normalized Run Time (Wall Clock)	Number of Solver Passes
Ogden	1.00	154
Two-Invariant Marlow	1.14	167

Table 1. Performance Comparison

5. Conclusion

The first-invariant Marlow model for the strain-energy density function in hyperelasticity in Abaqus is a special case of a more general two-invariant representation. Although the two-invariant representation is not a built-in Abaqus model, it can easily be implemented in the UHYPER user subroutine. In simulations, the more general representation will exactly reproduce the test data used in its definition.

The UHYPER subroutine described here is hard-coded to represent a particular material. It could easily be modified to represent a different material. All that is required is the redefinition of the cubic splines that are derived from the test data. Software could be developed that reads sets of test data and generates the appropriate UHYPER subroutines, complete with user-specified extrapolation behavior. Such a piece of software could conceivably lift the burden of having to sort amongst the many curve-fit models to find the most suitable one.

There are several details of the two-invariant Marlow model that need further attention. For example, there is its problematic behavior at the undeformed configuration. The uniaxial stress-strain curve will have a discontinuity in the slope at a stretch of 1. The UHYPER subroutine as it exists today works around this problem rather crudely. The author believes it is possible to resolve this problem while retaining the ability of the model to exactly represent the test data. Another lesser problem is the slight deviation of the response from the test data for a compressible material.

6. References

1. Dassault Systèmes, Abaqus Analysis User's Manual Version 6.7, 2007.
2. Marlow, Randall S., "A General First-Invariant Hyperelastic Constitutive Model", in Constitutive Models for Rubber III, A.A. Balkema, Lisse, 2003.