

Numerical Analysis of Punching Shear Failure of Reinforced Concrete Slabs

Karsten Winkler and Friedhelm Stangenberg

Ruhr-University Bochum, Universitätsstr. 150, 44780 Bochum, Germany
Institute for Reinforced and Prestressed Concrete Structures

Abstract: *Nearly no load bearing behaviour of reinforced concrete members allows such varied interpretations and complex discussions as the shear behaviour. Especially the three-dimensional problem of the punching shear failure of reinforced concrete members is internationally discussed. Nevertheless up to now, there is no unified design approach or even an overall accepted design model. Especially for large structural members, as they are commonly used in industrial structures and high-rise structures, the experimental background is missing. Because of extensive costs and very high test loads numerical solutions and FE-simulations are indispensable. By using Abaqus, finite-element simulations are currently performed at the institute for reinforced and prestressed concrete structures at the Ruhr-University Bochum. For the main investigation of the punching shear problem, the general conditions as well as the applicability of the constitutive law and the discretization have to be investigated, proved and verified first. The main goals of these preliminary investigations are the determination of influences of different element types, the discretization of the reinforcement and the size effects of the used material model. The concrete is modelled by 8-nodes or 20-nodes solid continuum elements. Regarding the concrete material behaviour, a non-linear user-defined material model based on the concrete damaged plasticity model is used. For the parametric study, small and medium-sized slabs are simulated for comparison and verification with experimental data to ensure realistic results of the large-sized structures. The article will report the latest results of these simulations and the special problems of simulating the punching shear failure of reinforced concrete slabs.*

Keywords: *Concrete, Constitutive Model, Damage, Experimental Verification, Failure, Fracture.*

1. Introduction

The actual German and European code provisions for the design and construction of reinforced concrete slabs against punching shear failure base on semi-empirically and statistically developed formulations. Although there are different models and approaches available that try to describe and explain the punching shear behaviour and failure, even nowadays formulations and design parameters for punching have to be found by experimental analysis.

In this context the empirical background is generally derived from small-scale tests. Usually, experimental data for punching shear failure can be found for slab thicknesses smaller than around 30.0 cm (CEB-FIP, 1993).

Especially industrial structures and high-rise buildings require extraordinary large concrete members, such as foundations and individual footings with very large effective depths up to 7.0 m (Klöcker, 2007). These large reinforced concrete members are mostly needed as footings for heavily-loaded single-columns or for transferring enormous big single loads. Regarding this type of reinforced concrete members, the experimental basis and background is completely missing. Particularly, this is up to two reasons:

On the one hand, the requirements and conditions for testing large slabs with high depths result in very high test loads. Therefore these experiments require tremendous, wide-spanned loading and test fields, to cause a pure and undisturbed punching shear failure. On the other hand, these experimental investigations result in extensive costs.

Especially brittle failure modes, as they can be found in shear and punching shear failure, are influenced by a pronounced size effect. This means, that with an increasing effective depth of the concrete member, the related shear failure load is decreasing. In this regard, former research of size effects in punching shear failure by (Bazant, 1987) resulted in a formulated size-effect law. Nevertheless, the size-effect law for punching shear failure could not be verified and validated doubtless caused by a lack of missing comparable experimental results.

Up to now, no generally admitted conclusions or predictions can be derived from the existing experimental results, because the used material properties as well as geometrical dimensions are differing in a wide range. The currently existing tests satisfy neither geometrical similarity nor similarity to the material characteristics.

The validation of the punching shear failure loads of thick slabs by testing down-scaled experimental slabs is according to the disregard of the size effect not admissible. Concerning this matter, the actual code provisions and parameters should be proved, analysed and validated for large concrete members, what makes numerical solutions and FE-simulations indispensable.

At the institute for reinforced and prestressed concrete structures at the Ruhr-University Bochum, finite-element simulations are currently performed with Abaqus. For the main investigation of the punching shear problem, the applicability as well as the general conditions of the constitutive law and the fineness of the mesh have to be investigated, proved and verified. Recent preliminary studies discover and analyse proper finite element models and parameters for the simulation of punching shear failure. The main goals of these investigations are the determination of influences of different element types, the discretization of the reinforcement and the size effects reflected by the used material model. Regarding the fact that the simulation results of thick slabs cannot be verified with full-scale experiments, the main attention is to represent as much as essential parameters close to reality.

In this context, the question arises, if and to what extent realistic and admissible declarations and results for the punching shear behaviour of (thick) reinforced concrete members can be achieved by finite element simulations.

2. Punching shear failure of reinforced concrete slabs

Punching shear failure of reinforced concrete slabs occurs when concentrated loads are initiated, causing very high shear- and axial stresses. At first, the combined stress performance leads to radial cracks, starting at the edge of the load application zone. Increasing the load causes tangential cracks around the load application zone. The failure state is reached when an inclined crack forms around the column. With a typical cylindrical punching failure cone (Figure 1) the column separates from the slab. Without shear reinforcement the punching shear failure performs in a brittle manner within the discontinuity region of the highly stressed slab at the column.

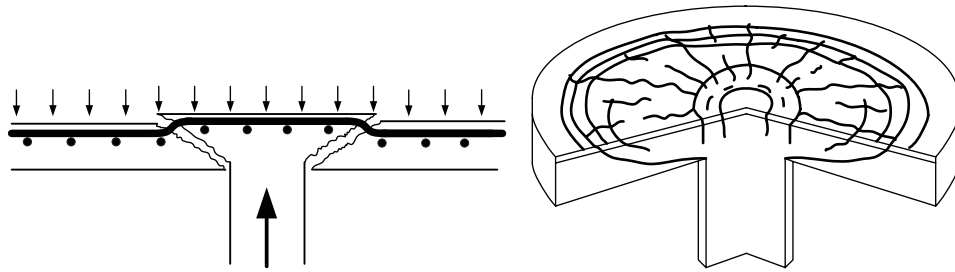


Figure 1. Cracking pattern and cylindrical cone of the punching shear failure.

Comparison of international code provisions leads to different approaches of calculating the punching shear failure, beginning with the approach of different shear crack angles as well as the consideration of different shear transferring mechanisms and resulting in varying values of maximum shear failure loads. Accordingly, different design concepts and design models are obvious.

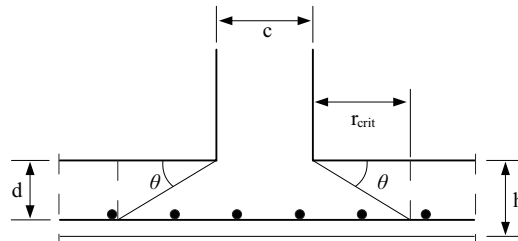


Figure 2. Design concept for punching shear failure according to DIN 1045-1 (DIN 1045-1, 2001).

With respect to the German code provisions of DIN 1045-1 (DIN 1045-1, 2001) the punching shear loads have to be proved in a control diameter r_{crit} (Figure 2). For non-shear reinforced concrete slabs this control diameter calculates to $r_{crit} = 1.5 \cdot d$, which corresponds to an assumed crack inclination of $\theta = 33.7^\circ$. In this regard, the design value of the punching shear resistance without shear reinforcement for normal strength concrete is formulated by:

$$v_{Rd,ct} = 0,14 \cdot \kappa \cdot (100 \cdot \rho_l \cdot f_{ck})^{1/3} \cdot d \quad (1)$$

The formulation considers the following parameters:

- The characteristic value of the concrete cylinder strength: f_{ck}
- The flexural reinforcement ratio: ρ_l
- The effective depth of the slab: d
- The assimilation factor for the size effect of the effective depth: κ

The maximum punching shear capacity $v_{Rd,max}$ depends on the concrete shear capacity:

$$v_{Rd,max} = 1.5 \cdot v_{Rd,ct} \quad (2)$$

The semi-empirical assimilation factor for the size effect κ was found by statistical analyses of shear tests of rectangular and profiled beams. Although there is no evidence that size effects of uniaxial shear for beams and punching shear are comparable or even the same, the German code assimilates the empirical evaluated size effect law of punching shear and uniaxial shear of beams by the same size factor. Referring to the German Code DIN 1045-1, the assimilation factor for the size effect κ reads by:

$$\kappa = 1 + \sqrt{\frac{200}{d[mm]}} \leq 2.0 \quad (3)$$

Comparing international code provisions, this assimilation of the size factor changes as well as any formulation for the assimilation of the size effect itself changes according to the number of statistical analysed test results.

Summarizing all approximate assumptions, the question arises, if the calculated punching shear failure load according to code provisions leads to unsafe, or even to an inefficient design of slabs with large effective depths.

For the conversion to the new European code provisions, there will be new modifications and formulations of the design concept against punching shear failure. Also international discussions on shear failure and shear behaviour of reinforced concrete members in general and especially on punching shear failure demonstrate the need for further research in these topics.

3. Finite Element model for punching

Essential for a finite element simulation of the punching shear problem of large structural concrete members is the detailed verification of discretization and material model using experimental test results. According to impracticably verification of simulation results with the structural behaviour of full-scale members, the resulting finite element model has to be sufficient to reasonably

simulate the punching shear behaviour of small-scale slabs as well as slabs with large effective depths.

In this regard, by preliminary studies the different possibilities of discretization and approximation of structural as well as material behaviour are researched, proved and analysed. Some of the main goals are the influence of the used material model, the type, geometrical shape and size of elements, the modality of discretization of the reinforcement and the discretization of support and load application area. The simulations are verified and evaluated by different types of experiments. Parametric studies figure out constraints and limits for material and discretization parameters.

According to these requirements, the finite element model is not finally completed and is modified and improved continuously.

3.1 Material model, functions and parameters for concrete

Although there are different possibilities to describe the complex, nonlinear material behaviour of concrete in Abaqus, suitable and admissible results for the simulation of the three-dimensional state of stress corresponding to the punching shear failure can only be derived from the elasto-plastic damage model “concrete damaged plasticity” (Abaqus, 2003).

Developed by (Lubliner, 1989) and elaborated by (Lee, 1998), the material model assumes a non-associated flow rate as well as isotropic damage. For the implementation of the concrete model, two types of material functions have to be defined. In this regard, stress-strain relations represent the uniaxial material behaviour under compressive and tensile loadings, which also includes cyclic un- and reloading. Among other things, suitable formulations for the stress-strain relations of concrete are given in (CEB-FIP, 1993; Pölling, 2000; Mark, 2006). Functions, which represent the evolution of the damage variables under compressive loadings d_c and tensile loadings d_t are shown in Equation 4 and Equation 5, respectively.

$$d_c = 1 - \frac{\sigma_c \cdot E_c^{-1}}{\varepsilon_c^{pl} \cdot (1/b_c - 1) + \sigma_c \cdot E_c^{-1}} \quad (4)$$

$$d_t = 1 - \frac{\sigma_t \cdot E_c^{-1}}{\varepsilon_t^{pl} \cdot (1/b_t - 1) + \sigma_t \cdot E_c^{-1}} \quad (5)$$

According to (CEB-FIP, 1993), the stress-strain relation behaviour of concrete under uniaxial compressive loading can be divided into three domains. As shown in Figure 3, the first section represents the linear-elastic branch, which can be formulated as a linear-elastic function of the secant modulus of elasticity E_c :

$$\sigma_c^{(1)} = E_c \cdot \varepsilon_c \quad (6)$$

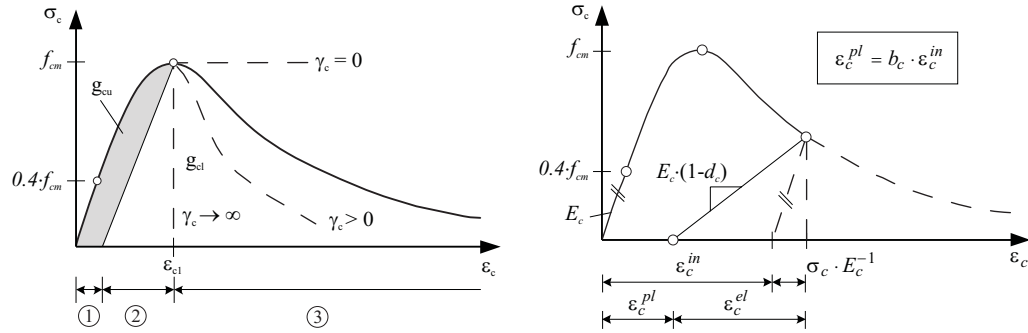


Figure 3. Stress-strain relation for uniaxial compressive loading.

While according to (CEB-FIP, 1993) the linear branch ends at $\sigma_c = 0.33 \cdot f_{cm}$, the stress-strain relations are negligibly modified and the first section is expanded up to $\sigma_c = 0.4 \cdot f_{cm}$. This allows the advantage of using the secant modulus as a material parameter that complies with the standards. Anymore, for verification of experiments with partial given material parameters, the modulus of elasticity can be taken from code approximations.

Equation 7 describes the ascending branch of the uniaxial stress-strain relation for a compression loading up to the peak load f_{cm} at the corresponding strain level ϵ_{cl} .

$$\sigma_c^{(2)} = \frac{E_{ci} \cdot \frac{\epsilon_c}{f_{cm}} - \left(\frac{\epsilon_c}{\epsilon_{cl}} \right)^2}{1 + \left(E_{ci} \cdot \frac{\epsilon_{cl}}{f_{cm}} - 2 \right) \cdot \frac{\epsilon_c}{\epsilon_{cl}}} \cdot f_{cm} \quad (7)$$

According to this, the modified parameter E_{ci} corresponds to the modulus of elasticity in Equation 7 (CEB-FIP, 1993; Mark, 2006) and can be calculated from:

$$E_{ci} = \frac{2}{3 \cdot E_c} \cdot \left(\frac{f_{cm}}{\epsilon_{cl}} \right)^2 - \frac{4}{3} \cdot \frac{f_{cm}}{\epsilon_{cl}} + \frac{5}{3} \cdot E_c \quad (8)$$

Section three in Figure 3 represents the post-peak branch and is described by Equation 9:

$$\sigma_c^{(3)} = \left(\frac{2 + \gamma_c \cdot f_{cm} \cdot \epsilon_{cl}}{2 \cdot f_{cm}} - \gamma_c \cdot \epsilon_c + \frac{\gamma_c \cdot \epsilon_c^2}{2 \cdot \epsilon_{cl}} \right)^{-1} \quad (9)$$

The post-peak behaviour depends on the descent function γ_c :

$$\gamma_c = \frac{\pi^2 \cdot f_{cm} \cdot \varepsilon_{cl}}{2 \cdot \left[g_{cl} - \frac{f_{cm}}{2} \left(\varepsilon_{cl} \cdot (1 - b_c) + \frac{b_c \cdot f_{cm}}{E_c} \right) \right]^2} > 0 \quad (10)$$

Basing on the assumption that the constant crushing energy G_{cl} (Pölling, 2000) is a material property, Equation 11 considers its dependency on the geometry of the tested or simulated specimen (Vonk, 1993; Van Mier, 1984) to almost eliminate mesh dependencies of the simulation results:

$$g_{cl} = \frac{G_{cl}}{l_c} \quad (11)$$

Herein l_c represents the characteristic length of the simulated or tested specimen. In the literature a wide differing range of values for the crushing energy G_{cl} can be found. In this regard, simulations of path controlled uniaxial compression tests are used to validate an admissible parameter for G_{cl} .

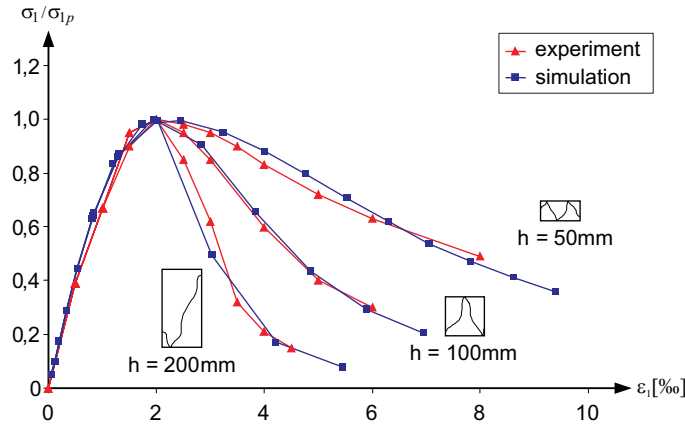


Figure 4. Comparison between simulation results and experimental data according to (Van Mier, 1984).

Figure 4 shows the simulation results of three concrete specimens with different heights and the same quadratic base area tested by (Van Mier, 1984). The simulations with Abaqus and the described material functions provide very good agreement with the given experimental values. The best approximation was found using a crushing energy of $G_{cl} = 19.0 \text{ kN} / \text{m}$.

This procedure is also used for the simulation of punching shear experiments. However, an uniaxial stress-strain relation for the used concrete including an utilizable part of the descending branch has to be given. Concerning this matter, for most publications of experimental data these information is unfortunately missing.

The description of the stress-strain relation for tensile loading is divided into two sections. Up to the maximum concrete tension strength, the linear part is calculated from:

$$\sigma_t^{(1)} = E_c \cdot \varepsilon_t \quad (12)$$

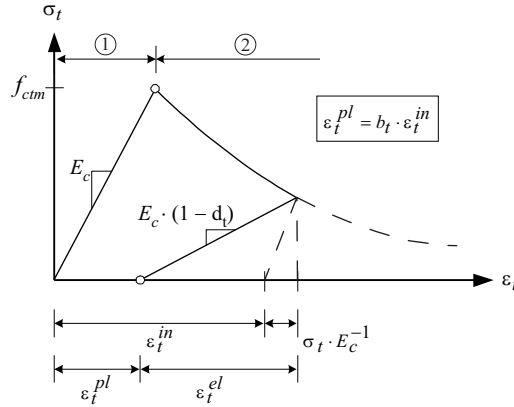


Figure 5. Stress-strain relation for uniaxial tension loading.

The descent branch of the stress-strain relation of concrete loaded in uniaxial tension can be derived from a stress-crack opening relation (Equation 13) according to (Hordijk, 1992), basing on the fictitious crack model of (Hillerborg, 1983).

$$\frac{\sigma_t(w)}{f_{ct}} = \left(1 + \left(c_1 \cdot \frac{w}{w_c} \right)^3 \right) \cdot e^{-c_2 \cdot \frac{w}{w_c}} - \frac{w}{w_c} \cdot (1 + c_1^3) \cdot e^{-c_2} \quad (13)$$

The free parameters could be experimentally determined to $c_1 = 3$ and $c_2 = 6.93$ (Hordijk, 1992). The function d_t (Equation 5) regulates the tension damage and includes the experimentally determined parameter b_t .

For multiaxial loadings, the material model was tested by simulating uniaxial, biaxial and triaxial strength experiments for normal strength concrete. The simulation results are in great accordance with the experimental data.

Figure 6 shows the comparison of the obtained simulation results for the described material functions and experimental results of biaxial loadings on concrete specimen by (Kupfer, 1973).

While the implemented “concrete damaged plasticity” material model reflects very successful the only-compression and only-tension ranges, the mixed-strength sectors seem to deviate from experimental relations and the published function definitions in (Abaqus, 2003). This is due to the fact that the sector of mixed compression-tension-stress is approximated by a linear function.

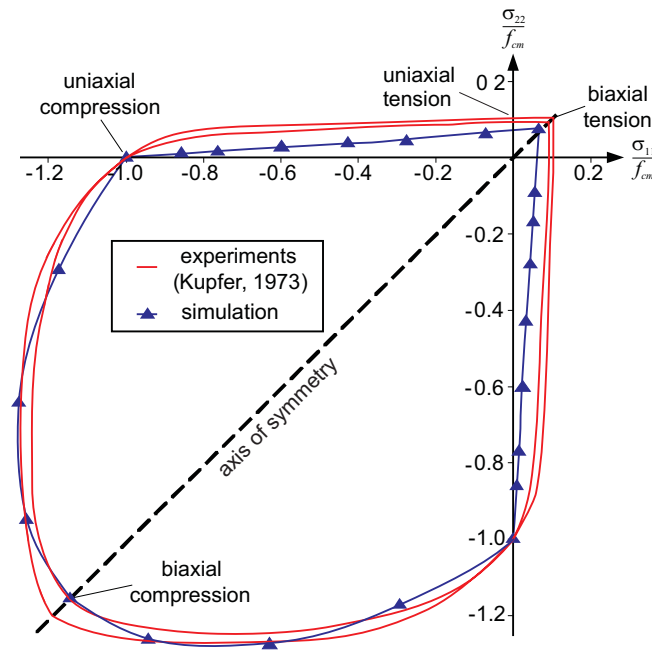


Figure 6. Biaxial strength of concrete: Comparison of experiments of (Kupfer, 1973) and simulation.

3.2 Material model for reinforcing steel

The uniaxial material behaviour of the reinforcing steel is modelled by a bilinear stress-strain relation. Therefore, the reinforcing steel material model only requires the yield strength f_y , the modulus of elasticity E_s as well as the ultimate strength f_u and the corresponding ultimate strain ε_u .

3.3 Mesh generation and discretization of reinforced concrete slabs

For the preliminary studies, a wide range of discretization variants has to be considered. Therefore, a parameterized discretization of the mesh has been created which allows not only the variation of geometrical parameters, but also enables the modification of element types, sizes and quantity.

For first verifications, rectangular slabs with rectangular load application zones and longitudinal supports have been considered. A typical test setup for rectangular concrete slabs according to punching failure is shown in Figure 7. The corresponding parameterized discretization is illustrated in Figure 8.

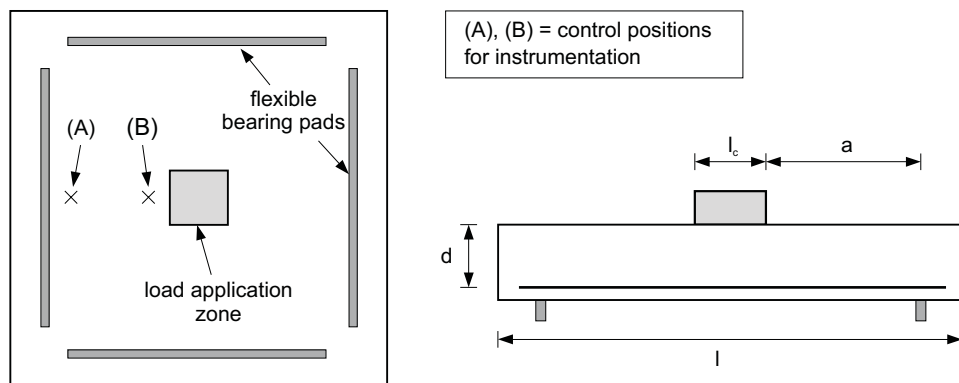


Figure 7. Horizontal projection and sectional drawing of typical test setups for punching.

Because of the very complex three-dimensional stress state within the punching area of the slab, three-dimensional solid (continuum) elements are used for the discretization of concrete elements. To satisfy the constraints of the used material model, the concrete mesh has to be almost cubic. Therefore, unintentional discretization and mesh influences on the crushing and fracture energy can be avoided (Pölling, 2000).

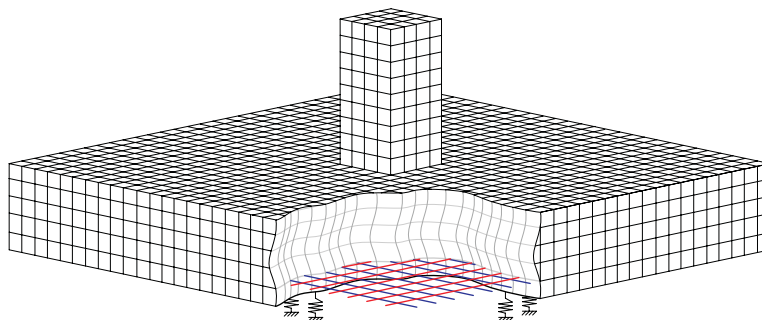


Figure 8. Finite element discretization of test setups for punching.

For preliminary analyses of the global bearing behaviour of the slabs, the whole system is discretized. Thus, also unsymmetrical test setups and excentric load or support cases can be analysed. For a more detailed analysis of inner stress states and crack propagation, and for application of high-order elements or finer discretizations, the symmetry will be utilized to shorten the calculation processes and to reduce data interpretation, of course.

The reinforcement is discretized as three-dimensional 8-node truss elements (T3D8). Preliminary studies show that for the discretization of bending reinforcement, there's no remarkable difference by using three-dimensional truss or beam elements.

To model the composite between concrete and reinforcing steel, the reinforcement is implemented as embedded elements in the concrete's host elements. Therefore, the assumed ideal bonding behaviour between concrete and reinforcing steel has to be taken into account, regarding experimental studies and verification results.

3.4 Support and load

The conditions and requirements of support and load application area are affecting the experimental as well as the simulation results. In experimental setups, neoprene bearing pads according to Figure 7 are used to reduce the reaction of constraints influenced by the support.

Within the simulation the vertical support of the reinforced concrete slabs is realised by "compression only" springs. The discretization as spring elements enables the simulation of different types of support, even linear or plane bedded support as well as simplified models for ground, as it affects the punching behaviour of footings, for example.

The column is discretized similar to the concrete mesh of the slab. Concerning this matter, also three-dimensional 8 node solid continuum elements are taken. The load application is realized either by surface loads, distributed element loads or single node loads. To analyse the influence of stiffness of the load application member, the material properties for the discretized column can be modified independently from the concrete's material of the slab.

Regarding these first studies, only rectangular columns or application zones are discretized, respectively.

4. Verification of the model

As a first verification and validation of the described and discretized finite element model, punching shear tests on reinforced concrete slabs done by (Li, 2000) have been simulated. Rectangular shaped concrete slabs of different heights, supported by longitudinal neoprene bearing pads have been tested until failure occurred. The geometrical and material parameters of the chosen slab and corresponding simulation parameters are shown in Table 1 and Table 2, respectively.

For the static simulation the "modified Riks method" is used to solve the nonlinear set of equations. Therefore, the method uses the arc-length to measure the progress of the solution, while loads and displacements are solved simultaneously.

To manage the nonlinear simulation, frequently long-winded variations of the solution and control parameters have to be performed. Concerning this matter, limitations of the arc-length increments as well as modifications in the time incrementation parameters often improve the convergence behaviour of the simulations.

Table 1. Geometrical and material parameters of the experiment (Li, 2000).

Parameter	Denotation
$l = 145.0 \text{ cm}$, $d = 20.0 \text{ cm}$	Side length and effective depth of the quadratic slab
$l_c = 20.0 \text{ cm}$	Side length of the quadratic column
$a = 52.5 \text{ cm}$	Length of shearing zone
$\rho_l = 0.83 \%$	Ratio of tension reinforcement
$f_{cm} = 39.4 \text{ MPa}$	Average concrete cylinder strength
$f_{ctm} = 3.27 \text{ MPa}$	Average concrete tension strength
$E_{cm} = 29425 \text{ MPa}$	Secant modulus of elasticity
$\varepsilon_{cl} = -2.13 \text{ ‰}$	Strain at f_{cm}
$f_y = 465 \text{ MPa}$	Yield strength of tension reinforcement
$\varepsilon_y = 2.3 \text{ ‰}$	Yield strain of tension reinforcement

Table 2. Simulation parameters.

Parameter	Denotation
$G_{cl} = 19 \text{ kN/m}$, $G_f = 0.195 \cdot w_c \cdot f_{ctm}$	Crushing and fracture energies
$b_c = 0.7$, $b_t = 0.1$	damage parameters according to (Pölling, 2000)
$w_c = 180 \mu\text{m}$	Maximum crack opening according to (Hordijk, 1992).
$\nu = 0.2$	Poisson's ratio
$\psi = 30^\circ$	Dilatation angle according to (Lee, 1998)
$\alpha_f = 1.16$	Ratio of biaxial to uniaxial compressive strength according to (Kupfer, 1973)
$K_c = \frac{2}{3}$	Second stress invariant ratio

In Figure 9 the load-displacement curvatures of simulation and experiment are given. Relative to the mean displacement of the column, the diagrams show the comparative displacements of the control positions (A) and (B) according to Figure 7 and corresponding to the described instrumentation of the experiments by (Li, 2000).

Comparing the maximum failure loads, the simulation leads to a quite small aberration of about 2 % to the experimental value. However, the general agreement with the experimental results is not satisfying. Especially non-proportional high strength up to the occurrence of first cracks and smaller displacements compared to the experimental values can be observed. Especially close to the failure load, the simulations run more and more instable. Concerning this matter, variation and modification of the control parameters for the “Static Riks”-solution as well as the time incrementation becomes more and more important.

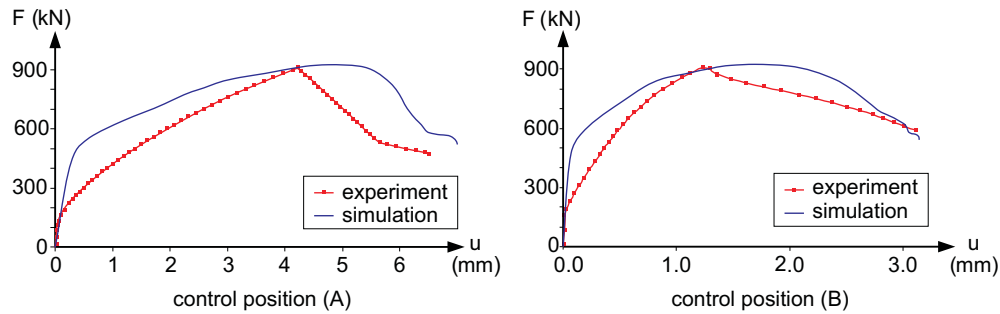


Figure 9. Load displacement curvature of experiment and simulation.

A closer look into the slab shows, that a general punching shear behaviour can be recognized. Illustrated in Figure 10, the typical inclined cracks – represented by the plastic strain results – agree quite well with the crack pattern of punching shear experiments (Figure 1). However, the inclination of the simulated cracks is too steep, but the typical punching shear cone can be identified.

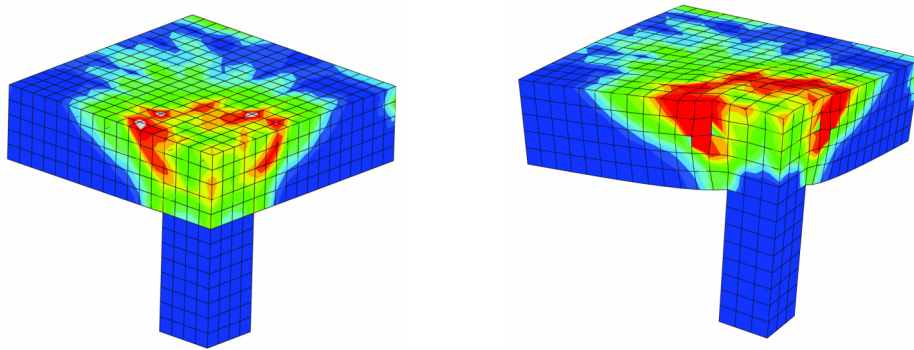


Figure 10. Crack pattern of the simulation of punching shear failure.

Comparing the crack pattern corresponding to the fineness of the discretized mesh, an intense dependence can be observed. Comparatively coarse discretizations cause numerical problems and do not lead to suitable results. Therefore, intensive analyses are actually performed.

5. Conclusions and further development

A finite element model to simulate the punching shear failure of rectangular reinforced concrete slabs is presented. The article reports about the preparation and verification of material model and mesh generation for simulating reinforced concrete slabs. The comparison of simulation of the concrete's material behaviour with test results shows very good approximation results. The verification of the presented finite element model is shown by simulation results of punching shear failure experiments. Therefore, a closer look at the comparison of theoretical assumptions,

experimental test results and simulation results shows that there are still questions left yet, like the mismatch of crack inclination, yielding of bending reinforcement or the differing displacement results. Furthermore, stabilization problems still occur during the simulation, especially for simulations of very brittle failure modes.

According to that, the preliminary studies will continue by determining problematic material and discretization parameters. This includes the verification with other test results and test setups of reinforced concrete slabs with smaller depths. By parametric studies the influence of discretization parameters will be investigated and analysed in the future.

Currently, the most problematic aspects are the lack of missing material data and setup facts. In this regard own experimental studies will complete the identification and analysis of needed simulation parameters.

Summarizing it must be considered, that further development and research is needed, before extrapolation to structural systems with bigger dimensions are possible. Missing test results and incomplete descriptions of test and material parameters cause the need of new experimental programs to validate and verify the described and used finite element model.

6. References

1. Abaqus Theory Manual, "Version 6.4," Simulia, USA, 2003.
2. Bazant, Z. P., and Z. Cao, "Size effect in punching shear failure of slabs," ACI Structural Journal, Vol. 84, pp. 44-53, 1987.
3. Bazant, Z. P., and B. H. Oh, "Crack Band Theory for fracture of concrete," Matériaux et Constructions, Vol. 16 (93), pp. 155-177, 1983.
4. Beutel, R., "Durchstanzen schubbewehrter Flachdecken im Bereich von Innenstützen," PHD-thesis, RWTH Aachen, 2003.
5. CEB-FIP, "Model Code 1990," Thomas Telford, London 1993.
6. CEB-FIP, "Punching of structural concrete slabs," bulletin 12, technical report, 2001.
7. DIN 1045-1, "Tragwerke aus Beton, Stahlbeton und Spannbeton," Beuth Verlag, Berlin, 2001.
8. DIN EN 1992-1-1, "Eurocode 2: Bemessung und Konstruktion von Stahlbeton- und Spannbetontragwerken," Beuth Verlag, Berlin, 2005.
9. Hegger, J., R. Beutel, A. Karakas, and F. Häusler, "Ersatz oder Teilersatz von Querkraftversuchen durch Finite-Elemente-Berechnungen – Abschlussbericht," DIBT, 2007.
10. Hegger, J., M. Ricker, B. Ulke, and M. Ziegler, "Untersuchungen zum Durchstanzverhalten von Stahlbetonfundamenten," Beton- und Stahlbetonbau, Vol. 101 (4), pp. 233-243, 2006.
11. Hillerborg, A., "Analysis of one single crack," Fracture Mechanics of Concrete, ed. Y. Wittmann, and F. H. Elsevier, pp. 223-249, Amsterdam, 1983.
12. Hordijk, D. A., "Tensile and tensile fatigue behaviour of concrete; experiments, modeling and analysis," Heron 37 (1), pp. 3-79, 1992.

13. Klöcker, T., and K. Winkler, "Durchstanzen massiger Fundamentplatten aus Stahlbeton," Festschrift Professor Stangenberg "25 Jahre in Forschung, Lehre und Praxis", Bochum, pp. 101-112, 2007.
14. Kupfer, H. B., and K. H. Gerstle, "Behaviour of Concrete under Biaxial Stresses," Journal of the Engineering Mechanics Division, Vol. 99, pp. 853-866, 1973.
15. Lee, J., and G. L. Fenves, "Plastic-damage model for cyclic loading of concrete structures," Journal of Engineering Mechanics, Vol. 124 (8), pp. 892-900, 1998.
16. Li, K. K. L., "Influence of Size on punching shear strength of concrete slabs," master thesis, McGill University, Montreal, Canada, 2000.
17. Linse, D., and H. Aschl, "Versuche zum Verhalten von Beton unter mehrachsiger Beanspruchung," test report, TU München, 1976.
18. Lubliner, J., J. Oliver, S. Oller, and E. Onate, "A plastic-damage model for concrete," International Journal for Solids and Structures, Vol. 25 (3), pp. 299-326, 1989.
19. Mark, P., "Zweiachsig durch Biegung und Querkkräfte beanspruchte Stahlbetonträger," Habilitationsschrift, Ruhr-Universität Bochum, 2006.
20. Pölling, R., "Eine praxisnahe, schädigungsorientierte Materialbeschreibung für Stahlbeton," Dissertation, Ruhr-Universität Bochum, 2000.
21. Reineck, K.H., D. A. Kuchma, K. S. Kim, and S. Marx, "Shear Database for Reinforced Concrete Members Without Shear Reinforcement," ACI Structural Journal, Vol. 100 (2), pp. 240-249, 2003.
22. Reinhardt, H.-W., "Maßstabseinfluss bei Schubversuchen im Licht der Bruchmechanik," Beton- und Stahlbeton Vol. 76 (1), pp. 19-21, 1981.
23. Sinha, B. P., K. H. Gerstle, and L. G. Tulin, "Stress-strain relations for concrete under cyclic loading," Journal of the ACI, Vol. 61 (2), pp. 195-211, 1964.
24. Van Mier, J. G. M., "Strain-softening of concrete under multiaxial loading conditions," PHD-thesis, Techn. Univ. Eindhoven, 1984.
25. Van Mier, J. G. M., H. W. Reinhardt, and B. W. Van der Vlugt, "Ergebnisse dreiachsiger verformungsgesteuerter Belastungsversuche an Beton," Bauingenieur, Vol. 62, pp. 353-361, 1987.
26. Vonk, R. A., "A micromechanical investigation of softening of concrete loaded in compression," Heron Vol. 38 (3), pp. 3-94, 1993.